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# NOTE

# A NOTE ON ARRAYS OF DOTS WITH DISTINCT SLOPES

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We prove that the maximum number of dots in an $n \times n$ array of dots with distinct slopes is at least $cn^{\frac{2}{3}}(\log n)^{-\frac{1}{3}}$ with $c>0$. This improves a previous result of $cn^{\frac{1}{2}}$. An upper bound is $O(n^{\frac{4}{5}})$.

## The Problem Formulation

Let $Z_n=\{1,2,\dots,n\}$. For $x$, $y\in Z_n^2$, $x\neq y$, denote by $S(x,y)=\frac{x_1-y_1}{x_2-y_2}$ the slope of the line through $x$ and $y$. For any line $l$ in $Z_n^2$, its slope can be written in the form $S(l)=\frac{u(l)}{v(l)}$, where $gcd(|u(l)|,\ |v(l)|)=1$ and $v(l)>0$. Define

$$H(l) = \max\{|u(l)|,\ |v(l)|\}. \tag{1}$$

If a subset $X$ of $Z_n^2$ satisfies the following condition:

$$\forall\{x,y\}\subset X \text{ and } \{z,w\}\subset X,\ \{x,y\}\neq\{z,w\}\rightarrow S(x,y)\neq S(z,w),$$

then we call $X$ an $n\times n$ array of dots with distinct slopes.

Define $\delta_n=\max\{|X|:X \text{ is an } n\times n \text{ array of dots with distinct slopes}\}$. It was proved in [1] that $cn^{\frac{1}{2}}\leq\delta_n\leq O(n^{\frac{4}{5}})$, where $c>0$. We improve the lower bound to $\delta_n\geq cn^{\frac{2}{3}}(\log n)^{-\frac{1}{3}}$.

## A Random Selection Theorem

Select a random array of dots $Y$ by letting

$$\forall x\in Z_n^2,\ Pr(x\in Y)=\frac{k}{n^2}.$$

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The selections for different $x$ are independent. Therefore, the probability of a specific array with $m$ dots being selected is

$$\left(\frac{k}{n^2}\right)^m \left(1-\frac{k}{n^2}\right)^{n^2-m}.$$

To construct a dot array with distinct slopes, we need to avoid the following two kinds of bad configurations: three points on a line or four points that form a trapezoid. A bad configuration can be eliminated by deleting one point from the array. Let $L$ be the number of triplets in $Z_n^2$ that are on a line and $T$ be the number of quadruplets in $Z_n^2$ that form a trapezoid. Then the expected number of triplets on a line in $Y$ is $k^3n^{-6}$ and the expected number of quadruplets that form a trapezoid in $Y$ is $Tk^4n^{-8}$. Deleting in average $Lk^3n^{-6}+Tk^4n^{-8}$ points from $Y$ to kill these bad configurations, the expected number of remaining points in $Y$ is

$$k-Lk^3n^{-6}-Tk^4n^{-8}.$$

Noting that

$$|\{l:H(l)=s\}|=O(ns^2),$$

and the number of points on a line $l$ is at most $O\left(\frac{n}{H(l)}\right)$, we obtain

(2) $$L\le O\left(\sum_{s+1}^{n} ns^2n^3s^{-3}\right)=O\left(n^4\log n\right),$$

and

(3) $$T\le O\left(\sum_{s=1}^{n} n^2s^3n^4s^{-4}\right)=O\left(n^6\log n\right).$$

Hence if $k^2<\frac{n^6}{3L}$ and $k^3<\frac{n^8}{3T}$, then the expected number of remaining points in $Y$ is at least $\frac{k}{3}$. The first inequality and (2) gives $k<O(n(\log n)^{-\frac{1}{2}})$ and the second inequality and (3) leads to $k<O\left(n^{\frac{2}{3}}(\log n)^{-\frac{1}{3}}\right)$, these two requirements are satisfied simultaneously. The result is proved.

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## References

[1] PAUL ERDŐS, RON L. GRAHAM, IMRE Z. RUZSA and HERBERT TAYLOR: Bounds for Arrays of Dots with Distinct Slopes or Length, *Combinatorica* **12** (1992), 39–44.

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